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M.A./M.SC. FIRST SEMESTER EXAMINATION, JANUARY 2012

FIRST YEAR

MATHEMATICS

Date : 13/01/2012

Time : 11am – 2pm

Paper : II

Full Marks : 60

All questions carry equal marks

Answer **any ten** questions:

1. (a) Show that the linear transformation $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ given by
$$T(f(x)) = 2f'(x) + \int_0^x 3f(t)\,dt$$
is not onto though one-one. 3

   (b) Let V be a finite dimensional vector space and W be a vector subspace of V. Then prove that $\dim \frac{V}{W} = \dim V - \dim W$, where $\frac{V}{W}$ denotes the quotient space of V with repect to the subspace W. 3

2. (a) Let A be the $m \times n$ matrix whose columns are $C_1, C_2, \ldots, C_n$ in the Euclidean m-space $\mathbb{R}^m$. Prove that the volume of the n-dimentional parallelepiped $P(A)$ determined by those vectors C_j's in $\mathbb{R}^m$ is given by vol (A)=$\sqrt{\det A^T A}$. 3

   (b) Consider the set of all those linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ that maps the line $y = x$ onto itself. Find, with justification, the dimension of this subspace of the vector space $M_{2\times 2}(\mathbb{R})$ 3

3. (a) Let F_3 be a finite field of 3 elements. How many 2×2 invertible matrices with entries from F_3 are there? Justify your answer. 4

   (b) 'All the eigenvalues of a unitary matrix over complex field are equal to ± 1'. Prove or disprove the statement. 2

4. (a) Let V^* denote the dual space of a vector space V with basis $\alpha = \{v_1, v_2, \ldots, v_n\}$. Defining the co-ordinate functions v_i^* with respect to the basis α, prove that the set $\alpha^* = \{v_1^*, v_2^*, \ldots, v_n^*\}$ of co-ordinate functions forms a basis for the dual space V^*, where for any $T \in V^*$, one has $T = \sum_{i=1}^n T(v_i)\, v_i^*$. 3

   (b) Let T be an orthogonal operator on $\mathbb{R}^2$ and let $A = [T]_\beta$ where β is the standard ordered basis for $\mathbb{R}^2$. Prove that exactly one of the following conditions are satisfied:

   a) T is a rotation and $det(A) = 1$;

   b) T is a reflection about a line through origin and $det(A) = -1$. 3

5. (a) The line through the origin in $\mathbb{R}^2$ making an angle θ with the positive direction of the x-axis is called the θ-line. Use geometry to find the matrices **P** of PROJECTION onto θ-line and **H** of REFLECTION onto θ-line. Justify geometrically that $H = 2P - I$ and use it to show that a reflection is its own inverse. 4

(b) Let V be a finite dimensional inner product space and E be the orthogonal projection of V on a subspace W. Prove that E is self-adjoint. 2

6. Consider the inner product space $P_2(\mathbb{R})$ with the inner product

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(t)g(t)\, dt.$$

Represent the polynomial $f(x) = 1 + 2x + 3x^2$ as a linear combination of the vectors in the orthonormal basis of $P_2(\mathbb{R})$ constructed out of its standard ordered basis. 3+3

7. (a) State and prove the Bessel's inequality in an inner product space. 1+2

(b) Let $f : V \to V$ be a rigid motion on a finite dimensional real inner product space V. Prove that there exist a unique orthogonal operator T on V and a unique translaion g on V such that $f = g \circ T$. 3

8. (a) For every invertible complex $n \times n$ matrix B, prove that there exist unique matrices N and U such that N is lower triangular with positive entries in the main diagonal and U is unitary. 5

(b) Prove that the rows of a unitary $n \times n$ matrix A over a field F form an orthonormal basis of F^n. 1

9. Let V be a finite dimentional inner product space over a field F and let g be a linear functional on V. Then prove that there exists a unique vector $y \in V$ such that $g(x) = \langle x, y \rangle$ for all $x \in V$. Prove further that y belongs to the orthogonal complement of the null space of g. Give a suitable counter example to justify that the result may fail to hold when V is not finite dimentional. 3+2+1

10. Verify that $A = \begin{bmatrix} 2 & 1-i \\ 1+i & 1 \end{bmatrix}$ is Hermitian. Diagonalize A by a unitary matrix U. 1+5

11. (a) Let $S : V \to W$ be a linear transformation, where $\alpha = \{v_1, v_2, \ldots, v_n\}$ and $\alpha = \{w_1, w_2, \ldots, w_m\}$ be respectively the bases of V and W. For the transpose $S^* : W^* \to V^*$ prove that

$$[S^*]_{\beta^*}^{\alpha^*} = ([S]_{\alpha}^{\beta})^T$$

where the symbols bear their usual meaning. 3

(b) If a 3×3 matrix A has eigen values $1, 2, 3$, what are the eigenvectors of

$$B = (A - I)(A - 2I)(A - 3I)?$$ 3

12. (a) V be a finite dimensional vector space. Prove that each basis of the dual space V^* is the dual of some basis of V. 3

(b) Let V be an inner product space, and let T be a linear operator on V. Then prove that, T is an orthogonal projection if an only if T has an adjoint T^* and $T^2 = T = T^*$. 3

13. (a) Consider the inner product space H of all continuous complex valued functions defined on $[0, 2\pi]$ under the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)}\, dt.$$

Show that $S = \{e^{int} : n \text{ is an integer}\}$ is an orthonormal set in H and compute the Fourier coefficients of $f(t) = t$ relative to S. 3+1

(b) Let V be a finite dimensional vector space over the field F and let W be a subspace of the vector space V. Then prove that $\dim W + \dim W^0 = \dim V$, where W^0 is the annihilator of W as usual. 2

14. (a) Let T be a linear operator on a finite dimensional inner product space V. Suppose that the characteristic polynomial of T splits. Then prove that there exists an orthonormal basis β for V such that the matrix $[T]_\beta$ is upper triangular. 4

(b) Let T be a normal operator on a finite dimensional complex inner product space. Then prove that there exists an orthonormal basis for V consisting of eigen vectors of T. 2

15. (a) If an upper triangular matrix T is normal, prove that it must be a diagonal matrix. 3

(b) Given that, the eigenvalues of the normal matrix $A = \begin{bmatrix} 0 & 0 & i \\ 0 & i & 0 \\ i & 0 & 0 \end{bmatrix}$ are $i, i, -i$ and corresponding orthonormal eigenvectors respectively are

$$u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad u_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

find the spectral decomposition of A. 3